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Extendible Pseudocomplex Structures*

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0. INTRODUCTION

Let M' be a complex manifold and let $M \subset M'$ be a relatively compact open subset whose boundary M_0 is a C^∞ submanifold of M'. The purpose of this paper is to investigate the question of extending pseudocomplex structures on M_0 sufficiently close to the one induced by the embedding of M_0 in M' to complex structures on M. It is established that the extension holds if the following conditions are satisfied:

I. $H^q(M, \wedge^n T'^* \otimes T'^* = 0$ for $q = n-2, n-1$, where $n = \dim_{\mathbf{C}} M = \dim_{\mathbf{C}} M'$; T' and T'^* are the holomorphic tangent and cotangent bundles over M', and $\wedge^n$ stands for the nth exterior algebra bundle, i.e., $K = \wedge^n T'^*$ is the canonical bundle. Furthermore, $K \otimes T'^*$ is the sheaf of germs of holomorphic sections of $K \otimes T'^*$, and $H^q(M, K \otimes T'^*)$ is the qth cohomology group of M with coefficients in $K \otimes T'^*$.

II. The Levi form of M_0 has at least two positive eigenvalues.

We observe that I holds if M is a Stein manifold, and if M' is Stein and M is strongly pseudoconvex, then both I and II hold. If one assumes II, then there are other conditions on M' which are of purely geometric nature and imply I. This can be seen as follows.

Let $C^{p,q}(\overline{M}, T'^*)$ be the space of all C^∞ T'^*-valued (p, q)-forms which are extendible to M', and let $\bar{\partial}^*$ be the formal adjoint of the Cauchy–Riemann operator $\bar{\partial}$ with respect to some metric g on M'. We denote by $\mathscr{H}^{p,q}$ the subspace of $C^{p,q}(\overline{M}, T'^*)$ consisting of the harmonic forms which satisfy the boundary conditions of the $\bar{\partial}$-Neumann problem. If II holds, then the

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theory developed in [2, 4] shows that $\mathscr{H}^{n,q}$ is isomorphic to $H^q(M, K \otimes T'^*)$ for $q = n - 1$, $n - 2$. By the Poincaré duality, $\mathscr{H}^{n,q}$ is isomorphic to $\mathscr{H}_d^{0,q} = \{u \mid \in C^{0,q}(\overline{M}, T')\ \bar{\partial}u = \bar{\partial}^*u = 0,\ tu = 0\}$ for $q = 1, 2$, where tu denotes the complex tangential part of u on M_0 (see Section 2). Hence condition I is equivalent to the following statement: there exists a constant $c_0 > 0$ such that

$$\| u \|^2 \leqslant c_0(\| \bar{\partial}u \|^2 + \| \bar{\partial}^*u \|^2) \tag{0.1}$$

for all $u \in C^{0,q}(\overline{M}, T'$ with $tu = 0$, $q = 1, 2$. Here $\| \quad \|$ is the L_2-norm with respect to g.

Now, in an earlier work of A. Andreotti and E. Vesentini (On deformations of discontinuous groups, *Acta Math.* (1964 , 112, it is established (see pp. 275–7 that if g is a Kähler–Einstein metric with sufficiently negative curvature, then for all $x \in M'$ and $u \in C^{0,q}(\overline{M}, T')$, $q = 1, 2$,

$$\langle u, u \rangle_x \leqslant c_0 \langle (\square - *^{-1} \square *)\, u, u \rangle_x \tag{0.2}$$

where $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, $*$ is the Hodge star-operator, and $\langle\ ,\ \rangle_x$ is the inner product at x.

Hence, by Stokes' theorem (0.2) implies (0.1) for all u with $tu = t\bar{\partial}^*u = 0$ and this is precisely what is needed for the proof of the main result of this paper (see Sections 2 and 3).

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1. Almost Pseudocomplex Structures and Integrability Conditions

Let $n = \dim_{\mathbb{C}} M$ and let $\mathbb{C}TM_0$ be the complexification of the real tangent bundle TM_0 of M_0.

Definition 1.1. An almost pseudocomplex structure on M_0 is given by a complex subbundle E'' of $\mathbb{C}TM_0$ of complex fiber dimension $n - 1$ such that $E' \cap E'' = \{0\}$ where $E' = \overline{E}''$.

Definition 1.2. E'' is integrable if for any two sections L and L' of E'' over an open set U of M_0, $[L, L']$ is also a section of E''.

There is a natural integrable almost pseudocomplex structure on M_0 given by ${}^\circ T'' = T'' \cap \mathbb{C}TM_0$, $T'' = \overline{T}'$. In general, if S is an almost complex structure on M, i.e., S is a complex subbundle of $\mathbb{C}TM$ of fiber complex dimension n such that $S \cap \bar{S} = \{0\}$, then $E'' = S \cap \mathbb{C}TM_0$ is an almost pseudocomplex structure on M_0. Moreover, if S is integrable, so is E''.

Let $\mathbb{C}T^*M_0$ be the complexified cotangent bundle of M_0 and let $(E'')^\perp \subset \mathbb{C}T^*M_0$ be the annihilator of E''.

DEFINITION 1.3. If $\theta^1,\ldots, \theta^n$ are differential forms of degree 1 on an open set U of M_0, then we say that they form a defining system of E'' over U if $\theta^j \in C^\infty(U, (E'')^\perp)$, $1 \leqslant j \leqslant n$, and, for each $p \in U$, $\{\theta_p{}^1,\ldots, \theta_p{}^n\}$ is a base of $(E''_p)^\perp$ where E''_p is the fiber of E'' over p.

PROPOSITION 1.4. *Let E'' be an almost pseudocomplex structure on M_0. Then the following conditions are equivalent*:

(a) *E'' is integrable.*

(b) *If $\{\theta^1,\ldots, \theta^n\}$ is a defining system over an open set U, $d\theta^j = \sum_k u_k{}^j \wedge \theta^k$, $1 \leqslant j \leqslant n$, for some differential form $u_k{}^j$ of degree* 1.

Proof. The assertion follows at once from the formula

$$2d\theta(L, L') = L \cdot \theta(L') - L' \cdot \theta(L) + \theta([L, L']) \tag{1.1}$$

which holds for any differential form θ of degree 1 and for all sections L, L' of $\mathbb{C}TM_0$. Q.E.D.

We now choose a real subbundle F of TM_0 of real fiber dimension 1 such that

$$\mathbb{C}TM_0 = {}^\circ T' \oplus {}^\circ T'' \oplus \mathbb{C}F, \qquad {}^\circ T' = \overline{{}^\circ T''}. \tag{1.2}$$

The existence of such F can be seen as follows. Since ${}^\circ T' \oplus {}^\circ T''$ is invariant under conjugation, there is a real vector subbundle ${}^\circ T$ of TM_0 such that ${}^\circ T' \oplus {}^\circ T'' = \mathbb{C}{}^\circ T$. By dimension consideration we have that ${}^\circ T$ is of real fiber codimension 1. Hence any supplementary vector subbundle F of ${}^\circ T$ in TM_0 satisfies (1.2). F is by no means unique but we pick one such F and fix it once and for all.

Let $\rho'\colon \mathbb{C}TM \to T'$ be the projection. It follows from (1.2) that $({}^\circ T' \oplus \mathbb{C}F) \cap (T'' \mid M_0) = \{0\}$ where $T'' \mid M_0$ is the restriction of T'' to M_0. Hence $\rho' \mid ({}^\circ T' \oplus \mathbb{C}F)$ is an isomorphism. We denote by

$$\tau : T' \mid M_0 \to {}^\circ T' \oplus \mathbb{C}F, \tag{1.3}$$

the inverse of $\rho' \mid ({}^\circ T' \oplus \mathbb{C}F)$. It is clear from the definition that τ is the identity on ${}^\circ T'$ and

$$\tau(\rho' X) = X \qquad \text{for all} \quad X \in \mathbb{C}F. \tag{1.4}$$

Let θ be a differential form of degree r.

DEFINITION 1.5. We say that θ is of type $(0, r)_b$ if $\theta(X_1, \ldots, X_r) = 0$ whenever any one of $X_1, \ldots, X_r$ is a section of $°T' \oplus \mathbb{C}F$.

DEFINITION 1.6. An almost pseudocomplex structure E'' on M_0 is of finite distance to $°T''$ if $\pi'' \mid E'' : E'' \to °T''$ is an isomorphism where π'': $\mathbb{C}TM_0 \to °T''$ is the projection with respect to the decomposition (1.2). It is easy to see that in this case we can write

$$E'' = \{X - \varphi_1(X) \mid X \in °T''\}. \tag{1.5}$$

Here $\varphi_1 : °T'' \to °T' \oplus \mathbb{C}F$ is a homomorphism defined by $\varphi_1 = -(id - \pi'') \circ (\pi'' \mid E'')^{-1}$. Let $\varphi = \tau^{-1} \circ \varphi_1 : °T'' \to T' \mid M_0$. Then φ is a $T' \mid M_0$-valued differential form of type $(0, 1)_b$. Hence we obtain the following

PROPOSITION 1.7. *If E'' is an almost pseudocomplex structure on M_0 of finite distance to $°T''$, then there exists a unique $T' \mid M_0$-valude differential form φ of type* $(0, 1)_b$ *such that*

$$E'' = \{X - \tau \circ \varphi(X) \mid X \in °T''\}. \tag{1.6}$$

Conversely, if $\pi' : \mathbb{C}TM_0 \to °T'$ is the projection with respect to the decomposition (1.2) and φ is a $T' \mid M_0$-valued differential form of type $(0, 1)_b$ with the property that at each point $x \in M_0$ the map $\bar{\sigma}_x \circ \bar{\sigma}_x : °T''_x \to °T''_x$ does not have eigenvalue 1, $\sigma = \pi' \circ \tau \circ \varphi$, then formula (1.6) defines an almost pseudocomplex structure on M_0.

DEFINITION 1.8. E'' defined by (1.6) is called the almost pseudocomplex structure determined by φ relative to $°T''$ and is denoted by $°T''_\varphi$.

Let $h : M' \to \mathbb{R}$ be a function defined as follows: $\mid h(p) \mid$ = geodesic distance from p to M_0, $h(p) > 0$ if $p \notin \overline{M}$, and $h(p) < 0$ if $p \in M$. Clearly there exists a neighborhood N of M_0 in M' such that h is of class C^∞ and $dh \neq 0$ in N. If $X \in \mathbb{C}TM_0$, $\langle X, dh \rangle = 0$ where $\langle X, dh \rangle$ is the evaluation of the differential form dh at X. Then $\langle \rho' X, dh \rangle = -\langle \rho'' X, dh \rangle$, and since h is real-valued $\langle \rho' X, dh \rangle$ is a purely imaginary number for $X \in TM_0$. We note that ρ'': $\mathbb{C}TM \to T''$ is the projection. On the other hand, since $\rho' F \cap °T' = \{0\}$, $\langle \rho' X, dh \rangle \neq 0$ for any nonzero $X \in F$. Thus, for each $x \in M_0$, there is a unique $X_x \in iF$ such that $\langle \rho' X_x, dh \rangle = 1$. We set $X_x = P_x' - P''_x$, $P''_x = \bar{P}_x'$, $P_x' \in T_x'$, $P''_x \in T''_x$. Hence we have sections P', P'' of $T' \mid M_0$, $T'' \mid M_0$, respectively, satisfying the following conditions

$$P' - P'' \subseteq iF, \qquad P' = \bar{P}'', \tag{1.7}$$

$$\langle P', dh \rangle = \langle P'', dh \rangle = 1. \tag{1.8}$$

We can extend P' (resp., P'') to a section of T' (resp., T''). Let $\tilde{U}$ be a coordinate neighborhood in N with coordinates $z = (z^1,..., z^n)$. We set $h_j = \partial h/\partial z^j$, $h_{\bar{j}} = \bar{h}_j = \partial h/\partial \bar{z}_j$, $P' = \sum_j p^j\partial/\partial z^j$, $P'' = \sum_j p^{\bar{j}}\partial/\partial \bar{z}^j$, $p^j = \bar{p}^j$. By (1.8) we have that

$$\sum_j p^j h_j = \sum_j p^{\bar{j}} h_{\bar{j}} = 1. \tag{1.9}$$

On $U = M_0 \cap \tilde{U}$ we define

$$Z_j = \partial/\partial z_j - h_j P', \qquad 1 \leqslant j \leqslant n. \tag{1.10}$$

It follows from (1.8) that $\langle Z_j, dh\rangle = 0$ and hence Z_j is a section of ${}^\circ T'$. Moreover, $Z_1,..., Z_n$ generate ${}^\circ T'$ and satisfy the relation

$$\sum_j p^j Z_j = 0. \tag{1.11}$$

If i: $U \to \tilde{U}$ is the injection, $d'h = \sum h_j dz^j$ and $d'h = \overline{d'h}$, we set

$$\bar{Z}^k = i^*\, d\bar{z}^k - p^{\bar{k}} i^*\, d''h = i^*\, d\bar{z}^k + p^{\bar{k}} i^*\, d'h, \qquad 1 \leqslant k \leqslant n. \tag{1.12}$$

We observe that $i^*dh = i^*(d'h + d''h) = di^*h = 0$ since $h = 0$ on M_0. The differential forms $\bar{Z}^i,..., \bar{Z}^n$ are of type $(0, 1)_b$ and generate $C^\infty(U, ({}^\circ T'')^*)$ where $({}^\circ T'')^*$ is the dual bundle of ${}^\circ T''$. By (1.12) we have that

$$\sum_k h_k \bar{Z}^k = 0. \tag{1.13}$$

Hence any differential form ψ of type $(0, r)_b$ can be written as

$$\psi = \sum \psi_{\bar{k}_1\cdots\bar{k}_r} \bar{Z}^{k_1} \wedge \cdots \wedge \bar{Z}^{k_r}, \tag{1.14}$$

$$\sum p^{\bar{l}} \psi_{\bar{l}\bar{k}_2\cdots\bar{k}_r} = 0. \tag{1.15}$$

Now, since $P' - P'' \in \mathbb{C}F$, we have by (1.4) that

$$\tau P' = P' - P''. \tag{1.16}$$

Since $Z_j \in {}^\circ T'$ and $\tau \mid {}^\circ T' = id$, $\tau(\partial/\partial z^j) = \tau(Z_j + h_j P') = Z_j + h_j P' - h_j P'' = \partial/\partial z^j - h_j P''$. We will write $\partial^\tau/\partial z^j$ instead of $\tau(\partial/\partial z^j)$. Thus we have that

$$\partial^\tau/\partial z^j = \partial/\partial z^j - h_j P'' \in \mathbb{C}TM_0, \qquad 1 \leqslant j \leqslant n. \tag{1.17}$$

If g is a differentiable function on $\tilde{U}$,

$$dg = \sum_k [(\partial g/\partial z^k)\, dz^k + (\partial g/\partial \bar{z}^k)\, d\bar{z}^k]$$
$$= \sum_k (\partial g/\partial \bar{z}^k)(d\bar{z}^k - \bar{p}^k\, d''h) + (P''g)\, dh + \sum_k (\partial g/\partial z^k - h_k P''g)\, dz^k.$$

Hence, if f is a differentiable function on U, we can write

$$df = \sum_k [(\partial f/\partial \bar{z}^k)\, \bar{Z}^k + (\partial^\tau f/\partial z^k)\, i^*\, dz^k]. \tag{1.18}$$

The operator $\sum_k (\partial/\partial \bar{z}^k)\, \bar{Z}^k$ is the boundary Cauchy–Riemann operator $\bar{\partial}_b$. Hence, (1.18) can be rewritten as

$$df = \bar{\partial}_b f + \sum_k (\partial^\tau f/\partial z^k)\, i^*\, dz^k. \tag{1.19}$$

We are now in a position to derive the integrability condition for the almost pseudocomplex structure ${}^\circ T''_\varphi$ in a convenient form. In terms of the local coordinates $(z^1,..., x^n)$ we have that $\varphi = \sum_k \varphi^k(\partial/\partial z^k)$ where φ^k is a scalar-valued differential form of type $(0, 1)_b$. It is then easy to check that

$$\theta^1 = i^*dz^1 + \varphi^1,..., \theta^n = i^*dz^n + \varphi^n \tag{1.20}$$

form a defining system of ${}^\circ T''_\varphi$. Hence, Proposition 1.4 implies that ${}^\circ T''_\varphi$ is integrable if and only if

$$d\varphi^k \equiv 0 \pmod{\theta^1,..., \theta^n}, \qquad 1 \leqslant k \leqslant n \tag{1.21}$$

for any coordinate system $(z^1,..., z^n)$.

First of all, we note that, by (1.19), for any differentiable function f on U we have $df \equiv \bar{\partial}_b f - \sum_k (\partial^\tau f/\partial z^k)\, \varphi^k \pmod{\theta^1,..., \theta^n}$. Hence

$$d\bar{Z}^k \equiv \sum_{j,l} [h_l \varphi^l \wedge (\bar{\partial}_b p^{\bar{k}} - (\partial^\tau p^{\bar{k}}/\partial z^j)\, \varphi^j) + p^{\bar{k}} \varphi^l \wedge (\bar{\partial}_b h_l - (\partial^\tau h_l/\partial z^j)\, \varphi^j)],$$
$$\operatorname{mod}(\theta^1,..., \theta^n). \tag{1.22}$$

Since $\varphi^k = \sum_l \varphi_{\bar{l}}{}^k \bar{Z}^l$ and $\sum_l p^{\bar{l}} \varphi_{\bar{l}}{}^k = 0$, (1.22) implies that

$$d\varphi^k = \sum_l [d\varphi_{\bar{l}}{}^k \wedge \bar{Z}^l + \varphi_{\bar{l}}{}^k\, d\bar{Z}^l]$$
$$\equiv \sum_{j,l} (\bar{\partial}_b \varphi_{\bar{l}}{}^k - (\partial^\tau \varphi_{\bar{l}}{}^k/\partial z^j)\, \varphi^j) \wedge \bar{Z}^l + \sum_{i,j,l} h_i \varphi^i \wedge \varphi_{\bar{l}}{}^k(\bar{\partial}_b p^{\bar{l}} - (\partial^\tau p^{\bar{l}}/\partial z^j)\, \varphi^j),$$
$$\operatorname{mod}(\theta^1,..., \theta^n). \tag{1.23}$$

We observe that $\bar\partial_b \bar Z^k = 0$ because $\bar Z^k - d\bar z^k$ is zero on ${}^\circ T'$. Then condition (1.23) can be written as

$$d\varphi^k \equiv \bar\partial_b \varphi^k - \sum_{j,l} (\partial^\tau \varphi_{\bar l}{}^k/\partial z^j)\, \varphi^j \wedge \bar Z^l + \sum_{i,j,l} h_i \varphi^i \wedge \varphi_{\bar l}{}^k (\bar\partial_b p^{\bar l} - (\partial^\tau p^{\bar l}/\partial z^j)\, \varphi^j), \quad \mathrm{mod}(\theta^1,\ldots,\theta^n). \tag{1.24}$$

Since T' is a holomorphic vector bundle, for any $T' \mid M_0$-valued differential form $\varphi = \sum_k \varphi^k \partial/\partial z^k$, $\sum_k (\bar\partial_b \varphi^k)(\partial/\partial z^k)$ is independent of the choice of z and hence represents a $T' \mid M_0$-valued form which we denote by $\bar\partial_b \varphi$. If we write the right-hand side of (1.24) as $\Phi^k \bmod (\theta^1,\ldots,\theta^n)$ it is not difficult to check that $\Phi = \sum_k \Phi^k(\partial/\partial z^k)$ is a well-defined $T' \mid M_0$-valued differential form of type $(0, 2)_b$. Now (1.21) and (1.24) imply the following

PROPOSITION 1.9. *The almost pseudocomplex structure ${}^\circ T''_\omega$ is integrable if and only if $\Phi = 0$.*

Let T''_ω be an almost complex structure on M determined by $\omega \in C^{0,1}(M, T')$.

PROPOSITION 1.10. $T''_\omega \cap \mathbb{C}TM_0 = {}^\circ T''_\varphi$ *if and only if*

$$\varphi_{\bar l}{}^j = \sum_{i,k} \omega_{\bar k}{}^j (\delta_{\bar l}{}^{\bar k} - h_{\bar l} p^{\bar k} + h_i \varphi_{\bar l}{}^i p^{\bar k}), \qquad 1 \leqslant l \leqslant n, \tag{1.25}$$

where

$$\varphi = \sum_{j,l} \varphi_{\bar l}{}^j \bar Z^l (\partial/\partial z^j) \qquad \text{and} \qquad \begin{aligned} \delta_{\bar l}{}^{\bar k} &= 1, && \text{if } k = l, \\ &= 0, && \text{if } k \neq l. \end{aligned}$$

Proof. It follows from Proposition 1.7 that ${}^\circ T''_\varphi$ is spanned by

$$\begin{aligned} Z_{\bar l}{}^\varphi = Z_{\bar l} - \sum_j \varphi_{\bar l}{}^j (\partial^\tau/\partial z^j) &= \partial/\partial \bar z^l - \sum_{i,j} (h_{\bar l} - h_i \varphi_{\bar l}{}^i)\, p^{\bar j} (\partial/\partial \bar z^j) - \sum_j \varphi_{\bar l}{}^j (\partial/\partial z^j) \\ &= \sum_{i,j} (\delta_{\bar l}{}^{\bar j} - h_{\bar l} p^{\bar j} + h_i \varphi_{\bar l}{}^i p^{\bar j})(\partial/\partial \bar z^j) - \sum_j \varphi_{\bar l}{}^j (\partial/\partial z^j), \qquad 1 \leqslant l \leqslant n. \end{aligned}$$

Since $T''_\omega = \{X - \omega(X) \mid X \in T''\}$, a vector $\sum_j \zeta^j(\partial/\partial z^j) + \sum_j \zeta^{\bar j}(\partial/\partial \bar z^j)$ belongs to T''_ω if and only if $\zeta^j = -\sum_k \omega_{\bar k}{}^j \zeta^{\bar k}$. Hence $Z_{\bar l}{}^\varphi \in T''_\omega$ if and only if (1.25) holds. Q.E.D.

Let $\psi = \sum \psi_{\bar k_1,\ldots,\bar k_r} d\bar z^{k_1} \wedge \cdots \wedge d\bar z^{k_r}$ be a differential form of type $(0, r)$ on M. Then we can write $i^*\psi = t\psi \bmod (i^* d'' h)$ where

$$t\psi = \sum \psi_{\bar k_1 \cdots \bar k_r} \bar Z^{k_1} \wedge \cdots \wedge \bar Z^{k_r}, \tag{1.26}$$

and we say that $t\psi$ is the complex tangential part of ψ. If, moreover, ψ is a differential form with values in a holomorphic vector bundle of rank m, then we can locally express ψ as an m-tuple of scalar-valued forms $(\psi^1,\ldots, \psi^m)$ and define $t\psi$ as $(t\psi^1,\ldots, t\psi^m)$.

We now assume that φ is ${}^\circ T'$-valued. This means that φ is a form of type $(0, 1)_b$ with values in $T' \mid M_0$ such that $\tau \circ \varphi = \varphi$. In terms of local coordinates $z = (z^1,\ldots, z^n)$ the last condition can be written as

$$\sum_i h_i \varphi_{\bar l}{}^i = 0, \qquad 1 \leqslant l \leqslant n. \tag{1.27}$$

In this case, if $\omega \in C^{0,1}(\overline{M}, T')$ such that T_ω^φ can be defined, then (1.13), (1.25), (1.26), and (1.27) imply that $T_\omega'' \cap \mathbb{C}TM_0 = {}^\circ T_\varphi''$ if and only if $\varphi = t\omega$. Let $\Omega = \bar\partial\omega - [\omega, \omega]$ where $\omega = \sum_{j,k} \omega_{\bar j}{}^k d\bar z^j(\partial/\partial z^k)$ and $[\omega, \omega] = \sum_{j,k,l} (\partial\omega_{\bar l}{}^k/\partial z^j)\, \omega^j \wedge d\bar z^l(\partial/\partial z^k)$.

PROPOSITION 1.11. *Let φ be a $T' \mid M_0$-valued differential form of type $(0, 1)_b$ satisfying* (1.27). *Let $\Phi = \sum_k \Phi^k(\partial/\partial z^k)$ be the form defined by* (1.24). *If $\omega \in C^{0,1}(\overline{M}, T')$ is such that $\varphi = t\omega$, then*

$$t\Omega = \Phi - \sum_{\alpha,j,k} \omega_{\bar\alpha}{}^k p^{\bar\alpha} h_j \Phi^j(\partial/\partial z^k). \tag{1.28}$$

Proof. Let $h_{\bar i,\bar j} = \partial h_{\bar i}/\partial \bar z^j$, $h_{\bar i,j} = \partial h_{\bar i}/\partial z^j$, $h_{i,\bar j} = \partial h_i/\partial \bar z^j$, and $h_{i,j} = \partial h_i/\partial z^j$.

Since $\varphi_{\bar j}{}^k = \omega_{\bar j}{}^k - h_{\bar j} \sum_l \omega_{\bar l}{}^k p^{\bar l}$, (1.13) and the fact that $\sum_{i,j} h_{\bar i,\bar j}\, \bar Z^i \wedge \bar Z^j = 0$ imply that $\sum_{i,j} (\partial\varphi_{\bar j}{}^k/\partial\bar z^i)\, \bar Z^i \wedge \bar Z^j = \sum_{i,j} (\partial\omega_{\bar j}{}^k/\partial \bar z^i)\, \bar Z^i \wedge \bar Z^j$. Thus $\bar\partial_b\varphi = t\bar\partial\omega$. Furthermore,

$$\sum_{j,l} (\partial^\tau \varphi_{\bar l}{}^k/\partial z^j)\, \varphi^j \wedge \bar Z^l = t[\omega, \omega]^k - \sum_\alpha \sum_{i,j,l} \omega_{\bar\alpha}{}^k p^{\bar\alpha} \omega_{\bar i}{}^j h_{\bar l,j} \bar Z^i \wedge \bar Z^l. \tag{1.29}$$

By assumption

$$\sum_j h_j \omega_{\bar i}{}^j = h_{\bar i} \sum_{\alpha,j} h_j \omega_{\bar\alpha}{}^j p^{\bar\alpha} \qquad \text{on } M_0, \qquad 1 \leqslant i \leqslant n. \tag{1.30}$$

We now apply the tangential differential operator $\partial/\partial\bar z^l - h_{\bar l}P'$ to the equality (1.30) and sum over i and l after multiplying by $\bar Z^i \wedge \bar Z^l$. Since $h_{j,\bar l} = h_{\bar l,j}$ we have that

$$\sum_{i,j,l} h_{\bar l,j} \omega_{\bar i}{}^j \bar Z^i \wedge \bar Z^l = - \sum_{i,j,l} h_j (\partial\omega_{\bar i}{}^j/\partial\bar z^l)\, \bar Z^i \wedge \bar Z^l = \sum_j h_j t\, \bar\partial\omega^j = \sum_j h_j\, \bar\partial_b \varphi^j. \tag{1.31}$$

On the other hand, condition (1.27) implies that $\Phi^j = \bar\partial_b\varphi^j - \sum_i (\partial^\tau \varphi_{\bar k}{}^j/\partial z^i)\, \varphi^i \wedge \bar Z^k$ and $\sum_j h_j(\partial^\tau\varphi_{\bar k}{}^j/\partial z^i) = -\sum_j \varphi_{\bar k}{}^j(\partial^\tau h_j/\partial z^i) = -\sum_j h_{j,i}\varphi_{\bar k}{}^j$. But $h_{i,j} = h_{j,i}$, so $\sum_{i,j} h_{i,j}\varphi^i \wedge \varphi^j = 0$. Hence

$$\sum_j h_j \Phi^j = \sum_j h_j\, \bar\partial_b \varphi^j. \tag{1.32}$$

Finally, we obtain the desired result by combining (1.29), (1.31), and (1.32).

Q.E.D.

2. A Noncoercive Boundary Value Problem

In this section we consider a boundary value problem which will play a crucial role in the extension of pseudocomplex structures. Our discussion will be based on the fundamentally important results that have been obtained in [2, 3].

Let $\bar\partial^*$ be the formal adjoint of $\bar\partial$ with respect to the metric g in M'. Let $\omega \in C^{0,1}(\overline{M}, T')$. We will assume that for some sufficiently large integer $k > 0$ the Sobolev k-norm $\| \omega \|_k$ is sufficiently small with respect to various absolute constants which will appear in the sequel and will be denoted by C. For $u, v \in C^{0,1}(\overline{M}, T')$ we consider the bilinear form

$$Q(u, v) = (\bar\partial u, \bar\partial v) + (\bar\partial^* u, \bar\partial^* v) - 2([\omega, u], \bar\partial v). \tag{2.1}$$

Let $\{U_\alpha\}$ be a finite set of holomorphic coordinate neighborhoods in M' which cover $\overline{M}$, and let $z_\alpha^1, \ldots, z_\alpha^n$ be holomorphic coordinates in U_α. For $u \in C^{p,q}(\overline{M}, T')$ we define the seminorms

$$\| u \|_z^2 = \sum_{i,\alpha,\nu} \int_{U_\alpha \cap \overline{M}} | \partial u_{IJ}^{(\alpha)\nu} / \partial z_\alpha^{\,i} |^2 \, dM,$$

$$\| u \|_{\bar z}^2 = \sum_{i,\alpha,\nu} \int_{U_\alpha \cap \overline{M}} | \partial u_{IJ}^{(\alpha)\nu} / \partial \bar z_\alpha^{\,i} |^2 \, dM,$$

where $u = \sum_{I,J,\nu} u_{IJ}^{(\alpha)\nu} dz_\alpha^{\,I} \wedge d\bar z_\alpha^{\,J} (\partial / \partial z_\alpha^{\,\nu})$ on U_α, $I = \{i_1 < \cdots < i_p\}$, $J = \{j_1 < \cdots < j_q\}$ and $dz_\alpha^{\,I} = dz_\alpha^{i_1} \wedge \cdots \wedge dz_\alpha^{i_p}$, $d\bar z_\alpha^{\,J} = d\bar z_\alpha^{j_1} \wedge \cdots \wedge d\bar z_\alpha^{j_q}$. We define $E^2(u) = \| u \|^2 + \int_{M_0} | u |^2 \, dS + \| u \|_{\bar z}^2$. Here $\| u \|^2$ is the L_2-norm on $C^{p,q}(\overline{M}, T')$ and dS is the volume element on M_0. Let νu be the complex normal component of u, i.e., νu consists of all terms in the local expression of $i^* u$ which are divisible by $i^* dh$, and let $\mathscr{D}^{p,q}(\overline{M}, T') = \{u \in C^{p,q}(\overline{M}, T') | \, \nu u = 0\}$. Assume $q > 0$, and for each point on M_0 the Levi form either has $n - q$ positive eigenvalues or $q + 1$ negative eigenvalues. Then the basic estimate of Kohn and Morrey holds (see [2, pp. 130–133; 4, pp. 458–459 and pp. 463–464]), i.e., for all $u \in \mathscr{D}^{p,q}$

$$E^2(u) \leqslant C(\| u \|^2 + \| \bar\partial u \|^2 + \| \bar\partial^* u \|^2). \tag{2.2}$$

Let $\mathfrak{B}^q = \{u \in C^{0,q}(\overline{M}, T') | \, tu = 0\}$. It is easily verified that $u \in \mathfrak{B}^q$ if and only if $* \# u \in \mathscr{D}^{n,n-q}(\overline{M}, T'^*)$ where T'^* is the dual bundle of T', $*$ is the Hodge star-operator, and $\#$ is defined as follows: if $u^{(\alpha)} = \sum_\nu' u^{(\alpha)\nu} (\partial / \partial z_\alpha^{\,\nu})$ is the local expression of u on the coordinate neighborhood U and $(g_{(\alpha)\mu\bar\nu})$ are the components of the metric tensor, then $(\# u)^{(\alpha)} = \sum_{\mu,\nu}' g_{(\alpha)\mu\bar\nu} \overline{u^{(\alpha)\nu}} \, dz_\alpha^{\,\mu}$.

Hence condition II of the Introduction implies that for all $u \in \mathfrak{B}^q$, $q = 1, 2$,

$$\| u \|^2 + \int_{M_0} | u |^2 \, dS + \| u \|_z^2 \leqslant C(\| u \|^2 + \| \bar{\partial} u \|^2 + \| \bar{\partial}^* u \|^2). \tag{2.3}$$

On the other hand, it follows from condition I in the Introduction that if $u \in \mathfrak{B}^q$ and $\bar{\partial} u = \bar{\partial}^* u = 0$, then $u = 0$, $q = 1, 2$. Since the norm $\mathbb{D}^2(u) = \mathbb{D}(u, u)$, $\mathbb{D}(u, v) = (u, v) + (\bar{\partial} u, \bar{\partial} v) + (\bar{\partial}^* u, \bar{\partial}^* v)$ is completely continuous with respect to the L_2-norm $\| \ \|$, we have that $\| u \|^2 \leqslant C(\| \bar{\partial} u \|^2 + \| \bar{\partial}^* u \|^2)$ for all $u \in \mathfrak{B}^q$ with $q = 1, 2$. This together with (2.3) implies that for all $u, v \in \mathfrak{B}^1$

$$| Q(u, u) | \geqslant C \| u \|^2, \tag{2.4}$$

$$C_1 \mathbb{D}^2(u) \leqslant | Q(u, u) | \leqslant C_2 \mathbb{D}^2(u), \tag{2.5}$$

$$C_1 Q_0(u, u) \leqslant | Q(u, u) | \leqslant C_2 Q_0(u, u), \qquad 2Q_0(u, v) = Q(u, v) + \overline{Q(v, u)}, \tag{2.6}$$

$$| Q'(u, v) |^2 \leqslant C Q_0(u, u) \, Q_0(v, v), \qquad 2(-1)^{1/2} \, Q'(u, v) = Q(u, v) - \overline{Q(v, u)}. \tag{2.7}$$

Let $\hat{\mathfrak{B}}^1$ be the completion of $\mathfrak{B}^1$ with respect to the norm $\mathbb{D}(u)$. Then the preceding inequalities imply that for each square-integrable T'-valued form f of type (0, 1) there exists a unique $u \in \hat{\mathfrak{B}}^1$ for which

$$Q(u, v) = (f, v) \qquad \text{for all } v \in \mathfrak{B}. \tag{2.8}$$

Now the space $\mathfrak{B}^1$ satisfies the requirements (a), (b), (c) given in [3], pp. 451–452. Moreover, conditions (i), (ii)′, (iii) of the same paper, pp. 452–453, hold for the bilinear form $Q(u, v)$. However, condition (ii) is not satisfied, i.e., the integrand of $Q'(u, v)$ contains products of first-order derivatives of u and v. Hence, we cannot directly conclude that if $f \in C^{0,1}(\overline{M}, T')$, then $u \in \mathfrak{B}$. We now proceed to show that one could overcome this difficulty and obtain the desired regularity result. First we observe that u is C^∞ in the interior of M because Q is strongly elliptic.

It is easy to see that each point of M_0 has a neighborhood U which admits a boundary coordinate system, i.e., a system $\{t^1, \ldots, t^{2n-1}, h\}$, where h is the function defined in Section 1, the t^i are C^∞ real-valued functions, and at every point of U, $\langle dt^i, dh \rangle = 0$, $1 \leqslant i \leqslant 2n - 1$. Let $\mathbb{R}_-^{2n} = \{(t^1, \ldots, t^{2n-1}, h) \mid h \leqslant 0\}$. If r is a C^∞ function with compact support in $\mathbb{R}_-^{2n}$ we define the partial Fourier transform

$$\tilde{r}(\xi, h) = \int_{\mathbb{R}^{2n-1}} \exp(-(-1)^{1/2} \, t \cdot \xi) \, r(t, h) \, dt, \tag{2.9}$$
$$\xi = (\xi^1, \ldots, \xi^{2n-1}), \qquad t = (t^1, \ldots, t^{2n-1}), \qquad t \cdot \xi = \sum_{i=1}^{2n-1} t^i \xi^i.$$

For real s the operator T_s is given by

$$(\widetilde{T_s r})(\xi, h) = (1 + |\xi|^2)^{s/2}\, \tilde{r}(\xi, h). \tag{2.10}$$

Then the tangential s-norm of r is

$$|||\, r\, |||_s = \|\, T_s r\, \|. \tag{2.11}$$

Let $D_j r = \partial r/\partial t^j$, $1 \leqslant j \leqslant 2n - 1$ be the tangential derivatives of r and $D_h = \partial r/\partial h$ be the normal derivative of r. Then the expression $|||\, Dr\, |||_s$ is given by

$$|||\, Dr\, |||_s^2 = |||\, D_h r\, |||_s^2 + |||\, r\, |||_{s+1}^2. \tag{2.12}$$

We may assume that $\langle \partial h, \partial h\rangle = 1$ at every point of U. We can then define a special moving frame on U to be a set $\zeta^1,\ldots, \zeta^n$ of (1, 0)-forms on U such that $\zeta^n = \partial h$ and $\langle \zeta^i, \zeta^j\rangle = \delta^{ij}$ at every point of U. If $u \in C^{p,q}(\overline{M}, T')$, then, on $U \cap \overline{M}$, $u = (u^1,\ldots, u^n)$ where $u^\alpha = \sum_{IJ} u^\alpha_{I\bar{J}} \zeta^{I\bar{J}}$, $\zeta^{I\bar{J}} = \zeta^{i_1} \wedge \cdots \wedge \zeta^{i_p} \wedge \bar{\zeta}^{j_1} \wedge \cdots \wedge \bar{\zeta}^{j_q}$, $1 \leqslant \alpha \leqslant n$. If r is a differential function on U, then $r_{\zeta^i} = \langle dr, \zeta^i\rangle$ and $r_{\bar{\zeta}^i} = \langle dr, \bar{\zeta}^i\rangle$. Hence $\bar{\partial} u^\alpha = \sum u^\alpha_{I\bar{J}\bar{\zeta}^i} \bar{\zeta}^i \wedge \zeta^{I\bar{J}} + \cdots$ and $\bar{\partial}^* u^\alpha = (-1)^{p+1} \sum_{m,I,H} \epsilon^{mH}_{\langle mH\rangle} u^\alpha_{I\langle mH\rangle \zeta^m} \zeta^{I\bar{H}} + \cdots$ where H runs through all the $(q - 1)$-tuples and $\langle mH\rangle$ is the ordered q-tuple consisting of m and H, $\epsilon^{mH}_{\langle mH\rangle}$ is the sign of the permutation taking mH into $\langle mH\rangle$, and the dots denote terms which do not contain differential components. It is clear that $tu = 0$ if and only if $u^\alpha_{I\bar{J}} = 0$ on M_0 whenever $n \notin J$. The seminorms $\|\, u\, \|_\zeta^2 = \sum_{i,\alpha,I,J} \|\, u^\alpha_{I\bar{J}\zeta^i}\|^2$ and $\|\, u\, \|_{\bar{\zeta}}^2 = \sum_{i,\alpha,I,J} \|\, u^\alpha_{I\bar{J}\bar{\zeta}^i}\|^2$ are equivalent to $\|\, u\, \|_z^2$ and $\|\, u\, \|_{\bar{z}}^2$, respectively. If the support of u lies in $U \cap \overline{M}$, then we set $|||\, u\, |||_s = \sum_{I,J,\nu} |||\, u^\nu_{I\bar{J}}\, |||_s$ and $|||\, Du\, |||_s = \sum_{I,J,\nu} |||\, Du^\nu_{I\bar{J}}\, |||_s$. We also have

$$|(u, v)| \leqslant |||\, u\, |||_s\, |||\, v\, |||_{-s}. \tag{2.13}$$

A linear operator A ; $C_0^\infty(\mathbb{R}^{2n}_-) \to C^\infty(\mathbb{R}^{2n}_-)$ is called an operator of tangential order ρ if for each real s there is a constant C_s such that $|||\, Ar\, |||_s \leqslant C_s\, |||\, r\, |||_{s+\rho}$ for all $r \in C_0^\infty(\mathbb{R}\_{}^n)$, i.e., for all C^∞ functions r with compact support on $\mathbb{R}^{2n}_-$. Let $\mathscr{S}(\mathbb{R}^{2n}_-)$ be the space of all C^∞ functions which together with their derivatives die out faster than any power of $|\, t\, | + |\, h\, |$ at infinity. It is well known that the operator defined by multiplication by such a function is of tangential order zero. Hence, if L is a first-order differential operator with coefficients in $\mathscr{S}(\mathbb{R}^{2n}_-)$, then

$$|||\, Lr\, |||_s \leqslant C_s\, |||\, Dr\, |||_s. \tag{2.14}$$

Let $\mathscr{A}_\rho$ be the set of all operators A of tangential order ρ such that $A = \zeta T_\rho \eta$ with $\zeta, \eta \in C_0^\infty(\mathbb{R}^{2n}_-)$. Each $A \in \mathscr{A}_\rho$ has the following properties: A and its adjoint A^* are of tangential order ρ; $A - A^*$ is of tangential order $\rho - 1$; $(Ar)(t, 0) = 0$ for all $r \in C_0^\infty(\mathbb{R}_-^{2n})$ with $r(t, 0) = 0$; if L is a first-order differential operator, then for each real s there is a constant C_s such that for all $r \in C_0^\infty(\mathbb{R}_-^{2n})$

$$|||[A, L]\, r\,|||_s \leqslant C_s\, |||\, Dr\, |||_{s+\rho-1}, \qquad [A, L] = AL - LA, \quad (2.15)$$

$$|||[A - A^*, L]\, r\,|||_s \leqslant C_s\, |||\, Dr\, |||_{s+\rho-2}, \quad (2.16)$$

$$|||[A, [A, L]]\, r\,|||_s \leqslant C_s\, |||\, Dr\, |||_{s+2\rho-2}. \quad (2.17)$$

If $u \in C^{p,q}(\overline{M}, T')$, ζ and η have their supports in $U \cap \overline{M}$ we define $Au = \sum Au_\alpha^{IJ}\zeta^{IJ}$, $A \in \mathscr{A}_\rho$. The regularity at the boundary follows from certain a priori estimates derived in [3, pp. 464–466 and pp. 471–472]. A close examination of the proof of these estimates shows that they hold if we have the following

LEMMA 2.1. *There is a constant C_ρ such that for all $A \in \mathscr{A}_\rho$ and all $u \in \mathfrak{B}^1$ with support in $U \cap \overline{M}$*

$$|\, Q(Au, Au)| \leqslant C_\rho(|\, Q(u, A^*Au)| + |||\, Du\, |||_{\rho-1}^2). \quad (2.18)$$

Proof. For forms u we denote by Lu the bracket $2[\omega, u]$. Since $Q(u, v) = D(u, v) - (Lu, \bar\partial v)$ is a first-order bilinear form, Lemma 3.1 of [3, p. 460], gives that

$$|\, 2Q(Au, Au) - Q(A^*Au, u) - Q(u, A^*Au)| \leqslant C_\rho'\, |||\, Du\, |||_{\rho-1}^2. \quad (2.19)$$

On the other hand, a straightforward calculation shows that

$$\begin{aligned}(LA^*Au, \bar\partial u) = {} & (Lu, \bar\partial A^*Au) + 2([L, A]\, u, \bar\partial Au) \\ & + 2(LAu, [A, \bar\partial]\, u) + ([L, A^* - A]\, Au, \bar\partial u) \\ & + (Lu, [A^* - A, \bar\partial]\, Au) + ([[L, A], A]\, u, \bar\partial u) \\ & + (Lu, [[A, \bar\partial], A]\, u) + ([L, A]\, u, (A^* - A)\, \bar\partial u) \\ & + ((A^* - A)\, Lu, [A, \bar\partial]\, u) + ([L, A]\, u, [A, \bar\partial]\, u) \\ & + ([A, L]\, u, [A, \bar\partial]\, u).\end{aligned}$$

Since $A \in \mathscr{A}_\rho$, $Au \in \mathfrak{B}^1$. Then (2.3), (2.5), and (2.15) with $s = 0$ imply that

$$\begin{aligned}([L, A]\, u, \bar\partial Au) &= O(|||\, Du\, |||_{\rho-1} \cdot |\, Q(Au, Au)|^{1/2}), \\ (LAu, [A, \bar\partial]\, u) &= O(|||\, Du\, |||_{\rho-1} \cdot |\, Q(Au, Au)|^{1/2}),\end{aligned} \quad (2.21)$$

where $B = O(R)$ if $|\, B\,| \leqslant \tilde{C}_\rho\, |\, R\,|$ for some constant $\tilde{C}_\rho$ depending on ρ.

Applying (2.13) with $s = 1 - \rho$, (2.14) with $s = \rho - 1$, and (2.16) with $s = 1 - \rho$ and $r = Au$, we have that $([L, A^* - A]\, Au, \bar{\partial}u) = O(|||\, DAu\, |||_{-1}\, |||\, Du\, |||_{\rho-1})$. By (2.12) and (2.15) $|||\, DAu\, |||_{-1}^2 = |||\, D_h Au\, |||_{-1}^2 + |||\, Au\, |||_0^2 = O(|||\, AD_h u\, |||_{-1}^2 + |||[D_h, A]\, u\, |||_{-1}^2 + |||\, u\, |||_\rho^2) = O(|||\, D_h u\, |||_{\rho-1}^2 + |||\, Du\, |||_{\rho-2}^2) = O(|||\, Du\, |||_{\rho-1}^2)$. Thus

$$([L, A^* - A]\, Au, \bar{\partial}u) = O(|||\, Du\, |||_{\rho-1}^2\,. \tag{2.22}$$

The same arguments can be used to estimate the fifth term on the right of (2.21). Hence

$$(Lu, [A^* - A, \bar{\partial}]\, Au) = O(|||\, Du\, |||_{\rho-1}^2). \tag{2.23}$$

Applying (2.13) with $s = 1 - \rho$, (2.14) with $s = \rho - 1$, and (2.17) with $s = 1 - \rho$ we obtain

$$\begin{aligned} ([[L, A], A]\, u, \bar{\partial}u) &= O(|||\, Du\, |||_{\rho-1}^2), \\ (Lu, [[A, \bar{\partial}], A]\, u) &= O(|||\, Du\, |||_{\rho-1}^2). \end{aligned} \tag{2.24}$$

Finally, Schwarz's inequality, (2.15), and the fact that $A - A^*$ is of tangential order $\rho - 1$ imply that each of the remaining terms on the right-hand side of (2.20) is $O(|||\, Du\, |||_{\rho-1}^2)$.

Now, (2.20), (2.21), (2.22), (2.23), (2.24), and the preceding remark give the inequality

$$\begin{aligned} &(LA^*Au, \bar{\partial}u) \\ &\quad = (Lu, \bar{\partial}A^*Au) + O(|||\, Du\, |||_{\rho-1} \cdot |\, Q(Au, Au\, |^{1/2}) + O(|||\, Du\, |||_{\rho-1}^2). \end{aligned} \tag{2.25}$$

If K stands for $\bar{\partial}$ or $\bar{\partial}^*$, then we have the relation

$$\begin{aligned} (KA^*Au, Ku) = {} & (Ku, KA^*Au) + 2\,\mathrm{Re}(KAu, [A, K]\, u) \\ & + 2(-1)^{1/2}\, \mathrm{Im}\{([K, A^* - A]\, Au, Ku) + ([[K, A], A]\, u, Ku) \\ & + ([K, A]\, u, (A^* - A)\, Ku) + ([K, A]\, u, [A, K]\, u) \\ & + ([K, A]\, u, KAu)\}. \end{aligned} \tag{2.26}$$

It is now clear that the arguments used in the derivation of (2.25) also imply

$$\mathbb{D}(A^*Au, u) = \mathbb{D}(u, A^*Au) + O(|||\, Du\, |||_{\rho-1} \cdot |\, Q(Au, Au\, |^{1/2}) + O(|||\, Du\, |||_{\rho-1}^2). \tag{2.27}$$

Hence, by combining (2.25) and (2.27) we have

$$Q(A^*Au, u)$$
$$= Q(u, A^*Au) + O(|||\, Du\, |||_{\rho-1} \cdot |\, Q(Au, Au)|^{1/2}) + O(|||\, Du\, |||_{\rho-1}^{2}). \quad (2.28)$$

The desired estimate (2.18) is obtained with the aid of (2.19) and (2.28).
Q.E.D.

If $\|\ \|_s$ denotes the Sobolev s-norm over $\overline{M}$, then the a priori estimates for the solution u of (2.8) also give the inequality

$$\|\, u\, \|_{s+1} \leqslant C_s \,\|f\|_s\,. \quad (2.29)$$

Remark 2.2. As a consequence of Stokes' theorem, the unique solution $u \in \mathfrak{B}^1$ of (2.8) has the property $t\bar{\partial}^*u = 0$. Assume that $f \in C^{0,1}(\overline{M}, T')$ and $\bar{\partial}^*f = 0$. Then another application of Stokes' theorem gives $\bar{\partial}\bar{\partial}^*u = 0$, and hence $\bar{\partial}^*u = 0$. Thus the equation $\bar{\partial}^*(\bar{\partial}u - 2[\omega, u]) = f$ has a unique solution $u \in \mathfrak{B}^1$ which satisfies (2.29)

Our next task is to investigate the dependence of the constant C_s in (2.29) on ω.

LEMMA 2.3. *For each $s > 1$ there exists a constant C_s such that for all r and a in $C_0^\infty(\mathbb{R}^{2n}_-)$*

$$\|[T_s\,, a]\, r\,\| \leqslant C_s(\|\, a\,\|_{n+2}\, |||\, r\, |||_{s-1} + \|\, a\,\|_{n+2+s}\, \|\, r\,\|). \quad (2.30)$$

Proof. $[T_s\,, a]\, r = T_s ar - aT_s r$

$$\widetilde{T_s ar}(\xi, h) = (1 + |\,\xi\,|^2)^{s/2} \int \tilde{a}(\eta, h)\, \tilde{r}(\xi - \eta, h)\, d\eta,$$
$$\widetilde{aT_s r}(\xi, h) = \int \tilde{a}(\eta, h)(1 + |\,\xi - \eta\,|^2)^{s/2}\, \tilde{r}(\xi - \eta, h)\, d\eta.$$

A routine computation shows that

$$|(1 + |\,\xi\,|^2)^{s/2} - (1 + |\,\xi - \eta\,|^2)^{s/2}\,|$$
$$\leqslant C_s((1 + |\,\xi - \eta\,|^2)^{s-1/2}\,|\,\eta\,| + (1 + |\,\eta\,|^2)^{s/2}). \quad (2.31)$$

Hence by the Schwarz inequality and (2.31) we have

$$|[\widetilde{T_s}\,, a]\, r(\xi, h)|^2$$
$$\leqslant C_s \Big\{ \int (1 + |\,\eta\,|^2)\,|\,\tilde{a}(\eta, h)|^2\, d\eta \cdot \int (1 + |\,\tau\,|^2)^{s-1}\,|\,\tilde{r}(\tau, h)|^2\, d\tau$$
$$+ \int (1 + |\,\eta\,|^2)^s\,|\,\tilde{a}(\eta, h)|^2\, d\eta \cdot \int |\,\tilde{r}(\tau, h)|^2\, d\tau \Big\}.$$

Now, let $\rho(h) = \int (1 + |\eta|^2) |\tilde{a}(\eta, h)|^2 \, d\eta$. Then $\rho(0) - \rho(h) = \int_h^0 \rho'(y) \, dy$. Since $D_h \tilde{a}(\eta, h) = \widetilde{D_h a}(\eta, h)$, it follows that $|\rho(h)| \leqslant |\rho(0)| + |||Da|||$. Here $\rho(0)$ is the Sobolev 1-norm on $\mathbb{R}^{2n-1}$ which we denote by $|a|_1$. By the Sobolev inequalities $|a|_1 + |||Da|||$ is bounded by $\|a\|_{n+2}$. A similar reasoning shows that

$$\int (1 + |\eta|^2)^s |\tilde{a}(\eta, h)|^2 \, d\eta \leqslant \|a\|_{n+2+s} . \tag{2.32}$$

We note that the last inequality takes care of the cases when s is an integer and s is a half-integer. This completes the proof. Q.E.D.

We now take a covering $\{V^\alpha\}$ of M and a refinement $\{U^\alpha\}$ such that $U^\alpha \subset\subset V^\alpha$. By identifying $V^\alpha \cap \bar{M}$ with an open set in $\mathbb{R}^{2n}$ we define the operator T_s^α on U^α which is the usual operator T_s multiplied by a function in $C_0^\infty(V^\alpha \cap \bar{M})$ and identically one on U^α. Then another way of defining the tangential s-norm for a form u is $|||u|||_s = \sum_\alpha \|T_s^\alpha \sigma^\alpha u\|^2$ where $(\sigma^\alpha)^2$ is a partition of unity with respect to the covering $\{U^\alpha\}$.

THEOREM 2.4. *Let u be the unique solution of* (2.8) *with* $tu = t\bar{\partial}^* u = 0$. *Then*

$$\|u\|_{m+1}^2 \leqslant C_m(\|f\|_m^2 + \|\omega\|_{k+m+1}^2 \|f\|_0^2). \tag{2.33}$$

Proof. We write

$$\begin{aligned}
&\|\bar{\partial} T_s^\alpha \sigma^\alpha u\|^2 + \|\bar{\partial}^* T_s^\alpha \sigma^\alpha u\|^2 - (L T_s^\alpha \sigma^\alpha u, \bar{\partial} T_s^\alpha \sigma^\alpha u) \\
&\quad = (T_s^\alpha \sigma^\alpha \Box_\omega u, T_s^\alpha \sigma^\alpha u) + ([\bar{\partial}, T_s^\alpha] \sigma^\alpha u, \bar{\partial} T_s^\alpha \sigma^\alpha u) + (T_s^\alpha [\bar{\partial}, \sigma^\alpha] u, \bar{\partial} T_s^\alpha \sigma^\alpha u) \\
&\qquad + ([\bar{\partial}^*, T_s^\alpha] \sigma^\alpha \bar{\partial} u, T_s^\alpha \sigma^\alpha u) + (T_s^\alpha [\bar{\partial}^*, \sigma^\alpha] \bar{\partial} u, T_s^\alpha \sigma^\alpha u) \\
&\qquad + ([\bar{\partial}^*, T_s^\alpha] \sigma^\alpha u, \bar{\partial}^* T_s^\alpha \sigma^\alpha u) + (T_s^\alpha [\bar{\partial}^*, \sigma^\alpha] u, \bar{\partial}^* T_s^\alpha \sigma^\alpha u) \\
&\qquad + ([\bar{\partial}, T_s^\alpha] \sigma^\alpha \bar{\partial}^* u, T_s^\alpha \sigma^\alpha u) + (T_s^\alpha [\bar{\partial}, \sigma^\alpha] \bar{\partial}^* u, T_s^\alpha \sigma^\alpha u) \\
&\qquad - ([L, T_s^\alpha] \sigma^\alpha u, \bar{\partial} T_s^\alpha \sigma^\alpha u) - (T_s^\alpha [L, \sigma^\alpha] u, \bar{\partial} T_s^\alpha \sigma^\alpha) \\
&\qquad - ([\bar{\partial}^*, T_s^\alpha] \sigma^\alpha L u, T_s^\alpha \sigma^\alpha u) - (T_s^\alpha [\bar{\partial}^*, \sigma^\alpha] L u, T_s \sigma^\alpha u)
\end{aligned} \tag{2.34}$$

where $\Box_\omega u = \Box u - 2\bar{\partial}^*[\omega, u] = \Box u - \bar{\partial}^* L u$.

We will denote by ϵ and $C(\epsilon)$ small and large constants, respectively. Using (2.15) we get that the second and sixth terms are bounded by $C(\epsilon)|||D\sigma^\alpha u|||_{s-1}^2 + \epsilon\{\|\bar{\partial} T_s^\alpha \sigma^\alpha u\|^2 + \|\bar{\partial}^* T_s^\alpha \sigma^\alpha u\|^2\}$. The third and seventh terms are obviously bounded by $C(\epsilon)|||u|||_s^2 + \epsilon\{\|\bar{\partial} T_s^\alpha \sigma^\alpha u\|^2 + \|\bar{\partial}^* T_s^\alpha \sigma^\alpha u\|^2\}$. Application of the commutator to the first half of the inner product and integration by parts imply that the fourth, fifth, eighth, and ninth terms imply that they are bounded by $C(\epsilon)\|u\|_s^2 + \epsilon\{\|\bar{\partial} T_s^\alpha \sigma^\alpha\|^2 + \|\bar{\partial}^* T_s^\alpha \sigma^\alpha u\|^2\}$. Since

$[T_s, aD_j]\, r = [T_s, a]\, D_j r$, we see by Lemma 2.3 that the tenth and the thirteenth terms are bounded by $C(\epsilon)\| \omega \|_{n+3}^2 \||| Du \|||_{s-1}^2 + \epsilon\{\| \bar{\partial} T_s^\alpha \sigma^\alpha u \|^2 + \|||\, LT_s^\alpha \sigma^\alpha u \|^2\} + C(\epsilon)\| \omega \|_{n+3+s}^2 \| u \|_1^2$. Finally, the eleventh term can obviously be bounded by $C(\epsilon)\| u \|_s^2 + \epsilon \| \bar{\partial} T_s^\alpha \sigma^\alpha u \|^2$, and another application of the commutator and integration by parts leads to bounding the twelfth term by $C(\epsilon)\| u \|_s^2 + \epsilon \| LT_s^\alpha \sigma^\alpha u \|^2$. At the beginning of this paragraph we assumed that $\| \omega \|_k$ is small with respect to various constants for sufficiently large k. We may now take, for example, $k > n + 3$. Then from (2.34) we obtain (by using (2.3) and condition I in the Introduction)

$$\| \bar{\partial} T_s^\alpha \sigma^\alpha u \|^2 + \| \bar{\partial}^* T_s^\alpha \sigma^\alpha u \|^2 \leqslant C(|(T_s^\alpha \sigma^\alpha f, T_s^\alpha \sigma^\alpha u)| + ||| Du |||_{s-1} + \| u \|_s + \| \omega \|_{k+s}^2 \| u \|_1^2). \quad (2.35)$$

On the other hand, the left-hand side of (2.35) is bounded from below by $||| DT_s^\alpha \sigma^\alpha u |||_{-1/2}$. Also, $|(T_s^\alpha \sigma^\alpha f, T_s^\alpha \sigma^\alpha u)| \leqslant ||| T_s^\alpha \sigma^\alpha f |||_{-1/2}\, ||| T_s^\alpha \sigma^\alpha u |||_{1/2} \leqslant C(\epsilon)\, ||| T_s^\alpha \sigma^\alpha f |||_{-1/2}^2 + \epsilon\, ||| T_s^\alpha \sigma^\alpha u |||_{1/2}^2$. Summing over α we have

$$||| Du |||_{s-1/2}^2 \leqslant C(||| f |||_{s-1/2}^2 + ||| Du |||_{s-1}^2 + \| u \|_s^2 + \| \omega \|_{k+s}^2 \| u \|_1^2). \quad (2.36)$$

Repeating the argument for the term $||| Du |||_{s+1}$ twice we get

$$||| u |||_{s+1/2}^2 \leqslant C(\| f \|_{s-1/2}^2 + \| u \|_s^2 + \| \omega \|_{k+s}^2 \| u \|_1^2). \quad (2.37)$$

We can now take $s = m + \frac{1}{2}$ where m is an integer. By a standard argument which is given in [3, pp. 465 and 466], and which uses the ellipticity of the quadratic form Q, we can also estimate all derivatives of u and obtain

$$\| u \|_{m+1}^2 \leqslant C(\| f \|_m^2 + \| u \|_{m+1/2}^2 + \| \omega \|_{k+s}^2 \| u \|_1^2). \quad (2.38)$$

Now, $\| u \|_{m+1/2}^2 \leqslant \epsilon \| u \|_{m+1}^2 + C(\epsilon)\| u \|_m^2$. Thus

$$\| u \|_{m+1}^2 \leqslant C(\| f \|_m^2 + \| u \|_m^2 + \| \omega \|_{k+m+1}^2 \| u \|_1^2). \quad (2.39)$$

Finally, by reduction we get (2.33). Q.E.D.

3. Extensions of Integrable Almost Pseudocomplex Structures

It is obvious that if S is a complex structure on M, $E'' = S \cap \mathbb{C}TM_0$ is an integrable almost pseudocomplex structure on M_0. We will show that if conditions I and II in the Introduction are satisfied, then any integrable $E'' \subset {}^\circ T' \oplus {}^\circ T''$ can be extended to a complex structure on M provided E'' is sufficiently close to ${}^\circ T''$.

For functions $r \in C_0^\infty(\mathbb{R}^d)$ and a real number p we define a smoothing operator that has been introduced by Nash in [7] by the formula

$$R'(p)\, r(x) = \int p^d \chi(py)\, r(x-y)\, dy \tag{3.1}$$

where $\chi(x)$ is a function whose Fourier transform $\hat{\chi}(\xi) \equiv 1$ for $|\xi| < \frac{1}{2}$ and is identically equal to zero for $|\xi| > 1$ and which is C^∞ in ξ. Then one can establish the following inequalities

$$\| R'(p)\, r \|_{k+m} \leqslant \text{const}\, p^m \| r \|_k , \qquad \| r - R'(p)\, r \|_k \leqslant \text{const}\, p^{-m} \| r \|_{k+m} . \tag{3.2}$$

Next we define Seeley's extension operator $E ; C_0^\infty(\mathbb{R}_-^k) \to C_0^\infty(\mathbb{R}^k)$ [see [8]) by

$$\begin{aligned} Er(x', y) &= r(x', y), && \text{for} \quad y \leqslant 0, \\ &= \sum_{k=0}^{\infty} a_k r(x', b_k y), && \text{for} \quad y > 0, \end{aligned} \tag{3.3}$$

where $x' = (x', \ldots, x^{k-1})$, and a_k and b_k are chosen so that $b_k = -2^k$ and $\sum_{k=0}^{\infty} a_k b_k{}^m = 1$ for every integer $m \geqslant 0$.

This operator has the property that it is bounded in the Sobolev norms over $\mathbb{R}_-^k$ and $\mathbb{R}^k$, respectively,

$$\| Er \|_m \leqslant \text{const}\, \| r \|_m . \tag{3.4}$$

If $\sqcap : C^\infty(\mathbb{R}^k) \to C^\infty \mathbb{R}_-^k)$ is the restriction, then $R''(p) = \sqcap \circ R'(p) \circ E$; $C_0^\infty(\mathbb{R}_-^k) \to C_0^\infty(\mathbb{R}_-^k)$ satisfies the same inequalities as (3.2) with the Sobolev norms taken over $\mathbb{R}_-^d$.

We now imbed the differentiable double $\tilde{M}$ of M in some euclidean space. Then the technique developed by Nash in [7] shows that the smoothing operator can be defined for arbitrary tensors on $\tilde{M}$. Furthermore, if U is a boundary coordinate neighborhood in $\tilde{M}$ and $\mathscr{T} = \{\mathscr{T}_{\gamma\delta\cdots}^{\alpha\beta\cdots}\}$ is a compactly-supported tensor in $U \cap \overline{M}$, then $E\mathscr{T} = \{E\mathscr{T}_{\gamma\delta\cdots}^{\alpha\beta\cdots}\}$ is a well-defined compactly-supported tensor in U. Conversely, if $\mathscr{T}$ is compactly-supported in U, then $\mathscr{T} = \{\sqcap \mathscr{T}_{\alpha\beta\cdots}^{\gamma\delta\cdots}\}$ is well-defined and compactly supported in $U \cap \overline{M}$. By using partition of unity we can thus conclude that for each real number p we have a linear map $R(p); C^{0,q}(\overline{M}, T') \to C^{0,q}(\overline{M}, T')$ such that the following property is satisfied; for any integers m, k there exists a constant $C_{m,k}$ such that for all $u \in C^{0,q}(M, T')$

$$\| R(p)\, u \|_{k+m} \leqslant C_{m,k}\, p^m \| u \|_k , \quad \| u - R(p)\, u \|_k \leqslant C_{m,k}\, p^{-m} \| u \|_{m+k} . \tag{3.5}$$

THEOREM 3.1. *Assume that M and M' satisfy conditions* I *and* II *in the Introduction. Let φ be a $T' \mid M_0$-valued C^∞ differential form of type $(0,1)_b$ with sufficiently small Sobolev k-norm $\mid \varphi \mid_k$ on M_0 for some sufficiently larger integer k, and $\mid \varphi \mid_\lambda \leqslant$ const p_0^λ for all $0 \leqslant \lambda \leqslant \lambda_0$, where $p_0 > 1$ is a sufficiently large real number and $\lambda_0 > k$ is a sufficiently large integer. Then there exists $\omega \in C^{0,1}(\overline{M}, T')$ such that $G(\omega) = 0$ and $t\omega = \varphi$ where $G(\omega) = \bar{\partial}^*(\bar{\partial}\omega - [\omega, \omega])$.*

Proof. We will follow the method of Moser given in [6]. If $\omega \in C^{0,1}(\overline{M}, T')$ we can write $\omega = \tilde{t}\omega + \tilde{\nu}\omega d''h$ and we observe that $i^*\tilde{t}\omega = t\omega$ and $\tilde{\nu}\omega \in C^{0,0}(\overline{M}, T')$. We set $i^*\tilde{\nu}\omega = \nu\omega$. Take $\omega_0 \in C^{0,1}(\overline{M}, T')$ such that $t\omega_0 = \varphi$, and $\| \omega_0 \|_\lambda \leqslant$ const $\mid \varphi \mid_\lambda$. We can regard ω_0 as an approximate solution of $G(\omega) = 0$ and observe that $\| G(\omega_0) \|_{k-2}$ is sufficiently small if the same is true for $\mid \varphi \mid_k$. The actual solution will be constructed as a limit of a sequence of approximate solutions $\omega_0, \omega_1, \ldots, \omega_j, \ldots$.

Let $p_0, p_1, \ldots, p_j, \ldots$ be a sequence of real numbers with p_0 as in the statement of the theorem and $p_{j+1} = p_j^{3/2}$.

For $\omega \in C^{0,1}(\overline{M}, T')$ we define $G'(\omega)(u) = \lim_{s\to 0} s^{-1}(G(\omega + su) - G(\omega)) = \bar{\partial}^*(\bar{\partial}u - 2[\omega, u])$.

Assume that ω_i, $0 \leqslant i \leqslant j$, have already been constructed such that

$$\| \omega_i \|_k \leqslant \eta, \qquad \text{for small } \eta, \tag{3.6}$$

$$\| \omega_i \|_\lambda \leqslant C p_i^\lambda, \qquad \text{for some constant } C \text{ and } 4 \leqslant \lambda \leqslant \lambda_0, \tag{3.7}$$

$$\omega_{i+1} = \omega_i + \tilde{t}u_i + R(p_{i+1})\, \tilde{\nu}u_i d''h, \tag{3.8}$$

where

$$G'(R(p_{i+1})\,\omega_i)\, u_i + G(R(p_{i+1})\,\omega_i) = 0, \qquad tu_i = 0, \bar{\partial}^* u_i = 0.$$

We remark that (3.6) and (3.7) hold for ω_0. The theory developed in Section 2 allows us to do (3.8) because of the first inequality of (3.5) with $m = 0$ and the inductive assumption (3.6). Observe that by construction $t\omega_i = \varphi$.

For any l we have

$$\| u_j \|_l = C_l(\| G(R(p_{j+1})\,\omega_j \|_{l-1} + \| R(p_{j+1})\,\omega_j \|_{l+k-2} \| G(R(p_{j+1})\,\omega_j \|_0). \tag{3.9}$$

(In (2.33) we have replaced the integer k by $k - 3$, i.e., we take $k > n + 6$ where $n = \dim_{\mathbb{C}} M$.)

We first note that $t\omega_{j+1} = t\omega_j = \varphi$. We now verify (3.7) for ω_{j+1}. Using (3.5), (3.9), and the inductive assumptions (3.6) and (3.7) we get

$$\begin{aligned} \| \tilde{\nu}\omega_{j+1} \|_\lambda &\leqslant \| \tilde{\nu}\omega_j \|_\lambda + \| R(p_{j+1})\, \tilde{\nu}u_j \|_\lambda \leqslant C p_j^{\lambda} + C_{\lambda,0}\, p_{j+1}^{\lambda} \| \tilde{\nu}u_j \|_0 \\ &\leqslant C p_j^\lambda + 2C_{\lambda,0} C_0 \eta \cdot p_{j+1}^\lambda . \end{aligned} \tag{3.10}$$

The last quantity is less than or equal to $\frac{1}{2}Cp_{j+1}^{\lambda}$ if p_0 is sufficiently large and η sufficiently small. Furthermore, we can establish the following. If, in general, $\square_\omega u = f$, with $tu = t\bar{\partial}^* u = 0$, then $\square_\omega \tilde{t}u = \tilde{t}f + N$, where N stands for terms involving the components of u, their first derivatives, and a linear combination of some of the second derivatives of the components of νu. Since $\tilde{t}u = 0$ on M_0, the coercive estimates obtained in [1] hold and we have

$$\| \tilde{t}u \|_\lambda \leqslant C_\lambda'(\| f \|_{\lambda-2} + \| N \|_{\lambda-2}) \qquad \text{for some constant } C_\lambda'. \tag{3.11}$$

We now apply this argument to $u = u_j$. Again with the aid of (3.5) and (3.9) we can establish that for some constant C_λ''

$$\begin{aligned} \| \tilde{t}u_j \|_\lambda \leqslant C_\lambda''(&\| G(R(p_{j+1}))\, \omega_j \|_{\lambda-2} \\ &+ \| R(p_{j+1})\, \omega_j \|_{\lambda+k-3} \| G(R(p_{j+1})\, \omega_j) \|_0 + \| \tilde{\nu}u_j \|_\lambda) \\ \leqslant C_\lambda''(&C_{\lambda-2,0} \| \omega_j \|_\lambda + C_{\lambda-3,k} p_{j+1}^{\lambda-3} \| \omega_j \|_k \| \omega_j \|_2 + \| \tilde{\nu}u_j \|_\lambda). \end{aligned} \tag{3.12}$$

The same arguments, applied to $\tilde{\nu}u_j$ give

$$\| \tilde{\nu}u_j \|_\lambda \leqslant C_\lambda(C_{\lambda,1} p_{j+1} \| \omega_j \|_\lambda + C_{\lambda-2,k} p_{j+1}^{\lambda-2} \| \omega_j \|_k \| \omega_j \|_2). \tag{3.13}$$

Since $\tilde{t}\omega_{j+1} = \tilde{t}\omega_j + \tilde{t}u_j$ we can conclude as before that if η is sufficiently small and p_0 is sufficiently large, than $\| \tilde{t}\omega_{j+1} \|_\lambda \leqslant \frac{1}{2}Cp_{j+1}^{\lambda}$ if $\lambda \geqslant 4$. This and (3.10) give (3.7) for $i = j + 1$.

Now, if $\omega = u + v$, $u, v \in C^{0,1}(\overline{M}, T')$, then $G(\omega) = G(u) + G(v) - 2\bar{\partial}^*[u, v]$. Having this remark in mind, an application of (3.5), (3.9), (3.12) yields the following chain of estimates (for simplicity we will denote by c_k constants, depending only on k and λ_0)

$$\begin{aligned} \| \omega_{j+1} - \omega_j \|_k &\leqslant c_k\{\| \tilde{t}u_j \|_k + p_{j+1} \| \tilde{\nu}u_j \|_{k-1}\} \\ &\leqslant c_k\{p_{j+1}^{k-2} \| G(R(p_{j+1}))\, \omega_j \|_{k-2} + p_{j+1}^{-\sigma} \| \tilde{\nu}u \|_{\sigma+k}\} \\ &\leqslant c_k\{p_{j+1}^{k-2}(\| G(\omega_j) \|_{k-2} + \| G(R(p_{j+1})\, \omega_j - \omega_j) \|_{k-2} \\ &\qquad + \| R(p_{j+1})\, \omega_j - \omega_j \|_k \| \omega_j \|_k + p_{j+1}^{-\sigma} \| \omega_j \|_{\sigma+2k-2}\} \\ &\leqslant c_k\{p_{j+1}^{k-2} \| G(\omega_j) \|_{k-2} + p_{j+1}^{-\sigma} \| \omega_j \|_{\sigma+2k-2}\}. \end{aligned} \tag{3.14}$$

We will determine σ shortly but for now the first condition we impose is $\sigma + 2k - 2 \leqslant \lambda_0$, Then

$$\| \omega_{j+1} - \omega_j \|_k \leqslant c_k\{p_{j+1}^{k-2} \| G(\omega_j) \|_{k-2} + p_{j+1}^{-\sigma} p_j^{\sigma+2k-2}\}. \tag{3.15}$$

In the next set of estimates we again let c_k be a constant depending only on k and λ. First of all, we have

$$\| G(\omega_j) \|_{k-2} \leqslant \| G(\omega_{j-1}) + G'(\omega_{j-1})(\omega_j - \omega_{j-1}) \|_{k-2} + c_k \| \omega_j - \omega_{j-1} \|_k^2. \tag{3.16}$$

We now have

$$\begin{aligned}
G(\omega_{j-1}) &= G(\omega_{j-1} - R(p_j)\,\omega_{j-1}) + G(R(p_j)\,\omega_{j-1}) \\
&\quad - 2\bar{\partial}^*[\omega_{j-1} - R(p_j)\,\omega_{j-1}, R(p_j)\,\omega_{j-1}]: \\
G'(\omega_{j-1})(\omega_j - \omega_{j-1}) &= \bar{\partial}^*(\bar{\partial}(\omega_j - \omega_{j-1}) - 2[\omega_{j-1}, \omega_j - \omega_{j-1}]) \\
&= \bar{\partial}^*(\bar{\partial}(\omega_j - \omega_{j-1}) - 2[\omega_{j-1} - R(p_j)\,\omega_{j-1}, \omega_j - \omega_{j-1}] \\
&\quad - 2[R(p_j)\,\omega_{j-1}, \omega_j - \omega_{j-1}]) \\
&= G'(R(p_j)\,\omega_{j-1})(\omega_j - \omega_{j-1}) - 2\bar{\partial}^*[\omega_{j-1} \\
&\quad - R(p_j)\,\omega_{j-1}, \omega_j - \omega_{j-1}].
\end{aligned}$$

These relations together with (3.5), (3.6), (3.7), (3.8), and (3.16) imply that

$$\begin{aligned}
&\| G(\omega_j)\|_{k-2} \\
&\quad \leqslant c_k\{\| \tilde{\nu}u_{j-1} - R(p_j)\,\tilde{\nu}u_{j-1}\|_k + \|\omega_{j-1} - R(p_j)\,\omega_{j-1}\|_k + \|\omega_j - \omega_{j-1}\|_k^2\} \\
&\quad \leqslant c_k\{p_j^{-\sigma}\|\tilde{\nu}u_{j-1}\|_{\sigma+k} + p_j^{-\sigma}\|\omega_{j-1}\|_{\sigma+k} + \|\omega_j - \omega_{j-1}\|_k^2\} \qquad (3.17) \\
&\quad \leqslant c_k\{p_j^{-\sigma}\|\omega_{j-1}\|_{\sigma+2k-2} + \|\omega_j - \omega_{j-1}\|_k^2\} \\
&\quad \leqslant c_k\{p_j^{-\sigma}p_{j-1}^{\sigma+2k-2} + \|\omega_j - \omega_{j-1}\|_k^2\}.
\end{aligned}$$

Combining (3.15) and (3.17) we have

$$\|\omega_{j+1} - \omega_j\|_k \leqslant c_k\{p_{j+1}^{k-2}p_j^{-\sigma}p_{j-1}^{\sigma+2k-2} + p_{j+1}^{k-2}\|\omega_j - \omega_{j-1}\|_k^2 + p_{j+1}^{-\sigma}p_j^{\sigma+2k-2}\}.$$

Set $\epsilon_{j+1} = p_{j+1}^{\mu}\|\omega_{j+1} - \omega_j\|_k$, where μ as well as σ are to be determined. Then the above inequality becomes

$$\epsilon_{j+1} \leqslant c_k\{p_{j+1}^{\mu+k-2}p_j^{-\sigma}p_{j-1}^{\sigma+2k-2} + p_{j+1}^{-\sigma+\mu}p_j^{\sigma+2k-2} + p_j^{-1/2(\mu-k)-3}\epsilon_j^2\}. \qquad (3.18)$$

We first choose $\mu > 0$ such that $-\frac{1}{2}(\mu - k) - 3 \leqslant 0$. With the choice of μ we now determine $\sigma > 0$ such that $\frac{3}{2}(\mu + k - 2) - \sigma + \frac{2}{3}(\sigma + k - 2) \leqslant -1$ and $-\frac{1}{2}\sigma + \frac{3}{2}\mu + 2k - 2 \leqslant -1$. If λ_0 is sufficiently large we will still have $\sigma + 2k - 2 \leqslant \lambda_0$. Hence (3.18) implies the inequality

$$\epsilon_{j+1} \leqslant c_k\{\epsilon_j^2 + p_j^{-1}\}. \qquad (3.19)$$

If one chooses $p_0 \leqslant 4c_k^2$ and $\epsilon_1 = p_1\|\omega_1 - \omega_0\|_k \leqslant 1/2c_k$ (which is certainly possible if η is sufficiently small) one finds that $\epsilon_i \leqslant 1/2c_k$ for $0 \leqslant i \leqslant j+1$. We can now verify (3.6) for ω_{j+1}.

$$\| \omega_{j+1} - \omega_0 \|_k \leqslant \sum_{\alpha=1}^{j+1} \| \omega_\alpha - \omega_{\alpha-1} \|_k \leqslant \frac{1}{2c_k} \sum_{\alpha=1}^{\infty} p_\alpha^{-\mu}. \tag{3.20}$$

Thus, $\| \omega_{j+1} \| \leqslant \| \omega_0 \|_k + 1/2c_k \sum_{\alpha=1}^{\infty} p^{-\mu} \leqslant \eta$ if $| \varphi |_k$ is sufficiently small and p_0 is sufficiently large. Thus the induction step is completed and we have the sequence $\omega_0, \omega_1, \dots, \omega_j, \dots$ of approximate solution. This is a Cauchy sequence in $H_k^{0,1}(\overline{M}, T')$, the completion of $C^{0,1}(\overline{M}, T')$ in the Sobolev k-norm, because

$$\| \omega_{j+m} - \omega_j \|_k \leqslant \sum_{\alpha=j+1}^{j+m} \| \omega_\alpha - \omega_{\alpha-1} \|_k \leqslant \frac{1}{2c_k} \sum_{\alpha=j+1}^{j+m} p_\alpha^{-\mu} \leqslant c' p_{j+1}^{-\mu}. \tag{3.21}$$

Let $\omega = \lim_{j\to\infty} \omega_j$ in $H_k^{0,1}(\overline{M}, T')$. By the Sobolev imbedding theorem we actually have that ω is of class C^{s_0} (i.e., the coefficients of ω have continuous derivatives of all order up to s_0) if $k > s_0 + n$. If k is sufficiently large, so is s_0 and (3.17) shows that $G(\omega) = 0$. Since $t\omega_j = \varphi$ for all j, we have $t\omega = \varphi$.

It remains to show that ω is actually of class C^∞, i.e., $\omega \in C^{0,1}(\overline{M}, T')$.

We will prove by induction on $s \geqslant \lambda_0$ the following statement: there exists a constant C_s such that

$$\| \omega_j \|_s \leqslant C_s p_j^s \qquad \text{for all sufficiently large } j. \tag*{$(3.22)_s$}$$

We note that (3.7) gives $(3.22)_s$ for all $s \leqslant \lambda_0$ and all j. In order to verify $(3.22)_{s+1}$ we observe that $\| \omega_j - \omega_0 \|_{s+1} \leqslant \sum_{\alpha=0}^{j-1} \| \omega_{\alpha+1} - \omega_\alpha \|_{s+1} \leqslant C_{s+1,0} \sum_{\alpha=0}^{j-1} \| u_\alpha \|_{s+1}$. An application of (3.9) with $l = s+1$ gives the inequality $\| u_\alpha \|_{s+1} \leqslant c_s' \| R(p_{\alpha+1}) \omega_\alpha \|_{s+k-1} \leqslant c_s' C_{s-1,k} p_{\alpha+1}^{s-1} \cdot \eta$ for some constant c_s'. Hence $\| \omega_j \|_{s+1} \leqslant \| \omega_0 \|_{s+1} + c_s' C_{s-1,k} C_{s+1,0} \cdot \eta \sum_{\alpha=0}^{j-1} p_{\alpha+1}^{s-1} \leqslant C_s p_j^{s+1}$ if j is sufficiently large.

We are now in a position to show that the sequence $\omega_0, \omega_1, \dots, \omega_j, \dots$ is a Cauchy sequence in every Sobolev s-norm. For this purpose we consider the following two statements

$$\| G(\omega_j) \|_{s-2} \leqslant C_s' p_j^{-2\mu}, \tag*{$(3.23)_s$}$$

$$\| \omega_{j+1} - \omega_j \|_{s+1} \leqslant C_s' p_{j+1}^{-\mu}, \tag*{$(3.24)_{s+1}$}$$

for all sufficiently large j and for some constant C_s'. We have already established $(3.24)_k$ and $(3.23)_k$ follows from (3.17) by taking σ sufficiently large which is possible if λ_0 is sufficiently large.

We first show that $(3.23)_s$ and $(3.24)_s$ imply $(3.24)_{s+1}$. Since $G(R(p_{j+1}) \omega_j) = G(\omega_j) - G(\omega_j - R(p_{j+1}) \omega_j) + 2\bar{\partial}^*[\omega_j - R(p_{j+1}) \omega_j, R(p_{j+1}) \omega_j]$, by applying (3.5), (3.9), and $(3.22)_s$ we obtain a chain of inequalities for all sufficiently large j with various constants denoted by d_s :

$$\begin{aligned}
\| \omega_{j+1} - \omega_j \|_{s+1} &\leqslant d_s p_{j+1}^{k+1}(1 + \| \omega_j \|_s) \| G(R(p_{j+1}) \omega_j \|_{s-1} \\
&\leqslant d_s p_{j+1}^{k+1}(1 + \| \omega_j \|_s)(\| G(\omega_j)\|_{s-1} + p_{j+1}^{-\tau} \| \omega_j \|_{s+\tau+1} \\
&\quad + p_{j+1}^{-\tau} \| \omega_j \|_{s+\tau+1} \| \omega_j \|_{s+1}) \\
&\leqslant d_s p_{j+1}^{k+1}(1 + \| \omega_j \|_s)(\| G(\omega_j) - R(p_{j+1}) G(\omega_j)\|_{s-1} \\
&\quad + \| R(p_{j+1}) G(\omega_j)\|_{s-1} + p_{j+1}^{-\tau} p_j^{s+\tau+1} + p_{j+1}^{-\tau} p_j^{2s+2+\tau}) \\
&\leqslant d_s p_{j+1}^{k+1}(1 + \| \omega_j \|_s)(p_{j+1} \| G(\omega_j)\|_{s-2} + p_{j+1}^{-\tau} p_j^{2s+2+\tau}).
\end{aligned} \tag{3.25}$$

Since we assume $(3.24)_s$ the sequence $\{\omega_j\}$ is bounded in the $\| \, \|_s$-norm. Thus we obtain $(3.24)_{s+1}$ with the aid of $(3.23)_s$ if τ is sufficiently large and μ has been chosen in advance to be sufficiently large with respect to k.

Next we will show that $(3.22)_s$ (for all s) and $(3.24)_{s+1}$ imply $(3.23)_{s+1}$. In order to do this we first write (3.16) with k replaced by $s + 1$. Second, we use the arguments preceding (3.17) and obtain

$$\begin{aligned}
\| G(\omega_j)\|_{s-1} &\leqslant d_s'\{\| \tilde{v} u_{j-1} - R(p_j) \tilde{v} u_{j-1} \|_{s+1} \\
&\quad + \| \omega_{j-1} - R(p_j) \omega_{j-1} \|_{s+1} \| \omega_j - \omega_{j-1} \|_{s+1} + \| \omega_j - \omega_{j-1} \|_{s+1}\} \\
&\leqslant d_s'\{p_j^{-\tau} \| \omega_{j-1} \|_{\tau+s+1+k-2} + p_j^{-\tau} \| \omega_{j-1} \|_{s+1+\tau} p_j^{-\mu} + p_j^{-2\mu}\} \\
&\leqslant d_s'\{p_j^{-\tau} p_{j-1}^{\tau+s+k-1} + p_j^{-\tau} p_{j-1}^{s+1+\tau} p_j^{-\mu} + p_j^{-2\mu}\}
\end{aligned} \tag{3.26}$$

for some constant d_s'. Hence $(3.23)_{s+1}$ holds if we choose τ sufficiently large. This completes the induction step and establishes $(3.24)_{s+1}$ for all s. But this means that the sequence $\{\omega_j\}$ is a Cauchy sequence in every Sobolev s-norm. Hence $\omega \in C^{0,1}(\overline{M}, T')$. Q.E.D.

The results in Section 1 indicate that if the almost pseudo-complex structure $E'' \subset {}^\circ T' \oplus {}^\circ T''$ is sufficiently close to ${}^\circ T''$, then $E'' = {}^\circ T''_\varphi$ for a unique $T' \mid M_0$-valued C^∞ differential form φ of type $(0, 1)_b$ satisfying condition (1.27). On the other hand, it is well known that an almost complex structure T''_ω on the even-dimensional manifold M induced by a T'-valued C^∞ form ω of type (0, 1) is a complex structure if and only if $\Omega = \bar{\partial}\omega - [\omega, \omega] = 0$. Therefore, in view of Proposition 1.10 we can formulate the extension problem stated at the beginning of this section as follows.

THEOREM 3.2. *Let $\{M, M'\}$ be a finite manifold, i.e., M is a relatively compact open subset of M' with C^∞ boundary M_0, such that conditions* I *and* II *are satisfied. Let φ be a $T' \mid M_0$-valued C^∞ differential form of type $(0, 1)_b$ with sufficiently small Sobolev norm $\mid \varphi \mid_k$ for some sufficiently large integer k.*

Assume that the almost pseudocomplex structure $^\circ T''_\varphi$ *defined by* φ *is integrable and* $^\circ T''_\varphi \subset {}^\circ T' \oplus {}^\circ T''$. *Then there exists* $\omega \in C^{0,1}(\overline{M}, T')$ *such that* $t\omega = \varphi$ *and* $\Omega = 0$.

Proof. By the previous theorem there is $\omega \in C^{0,1}(\overline{M}, T')$ such that $t\omega = \varphi$ and $\bar\partial^*\Omega = 0$. Furthermore, the properties of the Poisson bracket trivially imply that $\bar\partial\Omega = 2[\omega, \Omega]$. Since $t\Omega = 0$ by Proposition 1.11, we have by (2.3) the inequality

$$\| \bar\partial\Omega \| \leqslant 2 \| \omega \|_k \| \Omega \|_z \leqslant 2\eta C(\| \Omega \| + \| \bar\partial\Omega \|). \tag{3.27}$$

On the other hand, by condition I we find that

$$\| \Omega \| \leqslant C' \| \bar\partial\Omega \| \qquad \text{for some constant } C'. \tag{3.28}$$

Combining (3.27) and (3.28) we obtain

$$\| \bar\partial\Omega \| \leqslant 2\eta C(1 + C') \| \bar\partial\Omega \| . \tag{3.29}$$

Hence $\bar\partial\Omega = 0$ if η is sufficiently small. Then (3.28) gives $\Omega = 0$. Q.E.D.

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